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Evaluation of the basic systems of equations for turbulence measurements using the Monte Carlo technique

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A numerical experiment has been carried out to evaluate two of the methods available for finding the time-averaged mean velocity and the Reynolds stresses of a turbulent flow field using hot wires. The conventional method is based on the series expansion of the response equation, subsequent truncation of the series and time averaging. The improved method is based on squaring and time averaging without neglecting any terms. The method adopted to evaluate these two methods is based on the Monte Carlo simulation of a pseudo turbulent flow field using random-number generators and the corresponding hot-wire response, for a prescribed set of conditions, by assuming an appropriate model for the hot-wire response. The simulated hot-wire response and the calibration constants are then perturbed about their mean values to study the effects of errors in these quantities. The perturbed response is used to compute the time-averaged flow field by the two methods. The deviation of these values from the generated pseudo values, averaged over large number of trials, is used as the criterion to evaluate the methods. This procedure is also used to estimate the errors due to truncation in the conventional method, to study the effect of turbulence-intensity levels and to study the effects of measurement errors. The results indicate that the choice of the method for determining the time-averaged quantities should be based on the turbulence-intensity level and the measurement errors likely to be encountered. The conventional method yields reliable mean-velocity results for turbulence intensities as high as 50 % with second-order turbulence correction. If measurement errors are within reasonable limits and the turbulence level is below 20 %, the conventional method yields reliable results for Reynolds stresses. The improved method should be used to determine the time-averaged flow field for turbulence intensity above 40–50 %. The error in the yaw sensitivity parameter k has an insignificant effect on the mean velocity and Reynolds stresses computed by both methods. By accurately determining the sensitivity s of the hot wire, the accuracy of the measured mean velocity and Reynolds stresses can be improved significantly. An improved method of carrying out the uncertainty analysis for measurements, based on the Monte Carlo technique, has also been outlined.

1. Introduction

The basic problem in using hot-wire anemometry for turbulence measurements is in developing the response equation relating the measured hot-wire voltage to the velocity components. The common approach to this problem has been to use the empirical heat-transfer law put forth by King (1914) in the form

$$E_{NL}^2 = A + BU_{eff}^n. \tag{1}$$

Here E_{NL} is the anemometer nonlinear output voltage, U_{eff} is the effective cooling velocity and A, B and n are the calibration constants. The use of this equation raises problems due to the nonlinearity of the equation, which introduces errors in measurements of highly turbulent flow fields and due to the dependence of the constants B and n on the velocity. In practice, to overcome these problems linearizing circuits are generally used to reduce the form of the equation to

$$E = sU_{eff}, \tag{2}$$

where E and s are the linearized voltage output and calibration constant respectively. However, the use of commercially available linearizers requires an estimate of the constants A, B and n beforehand. The next part of the problem is to arrive at an analytical expression for U_{eff} in terms of the components of velocity in the wire-oriented coordinates. This involves knowledge of the directional sensitivities of the hot wire as well. Though there are several expressions in use for U_{eff}, the most commonly used one is due to Jorgensen (1971) and is given by

$$U_{eff} = [U_N^2 + k^2U_T^2 + h^2U_{BN}^2]^{\frac{1}{2}}, \tag{3}$$

where U_N, U_T and U_{BN} are the normal, tangential and binormal components of the velocity with respect to the wire and k and h are the yaw and pitch sensitivities of the hot wire. In a turbulent flow, the components U_N, U_T and U_{BN} can be written in terms of the mean and fluctuating velocity components in the three orthogonal directions, the yaw angle (α) and the pitch angle (ϕ). U_{eff} would then represent the instantaneous effective cooling velocity and E the instantaneous voltage output. The constants involved (s, k and h) are generally obtained from calibrations.

Essentially there are two methods in use for the measurement of mean velocities and Reynolds stresses of a turbulent flow field. In the first method, a 3-sensor probe is employed together with online data processing of the instantaneous flow field. In principle the method involves no further assumption other than that involved in arriving at (2) and (3). Implementing the method, however, is very expensive and hence beyond the reach of most of the hot-wire users. Further, instantaneous flow-field data is not in a very usable form for an engineer. The other commonly used method involves time averaging the response equation to derive expressions for the mean velocities and Reynolds stresses in terms of the mean and r.m.s. voltages of the hot wire. This method can be used in conjunction with a single wire, a two-wire sensor or a triple-wire sensor and a digital or analog data processing system. This paper is concerned only with the time-averaging scheme.

The time-averaging technique can in turn be carried out in two different ways. From (2) and (3), we can see that the hot-wire output is directly proportional to the effective velocity, which in turn is composed of the vector components of the velocity field. This implies that the components occur under a square-root sign and a simple time averaging of such an expression would lead to no solution of the problem on hand. Hence to derive the required expressions in the conventional method (see Hinze 1959; Champagne & Sleicher 1967), the equation for U_{eff} is first expanded in a series. To make the problem tractable the series is truncated assuming that the third- and higher-order terms are negligible compared with the second-order terms.

The resulting response equation, after truncation, is then time averaged. Manipulation of this equation then leads to a closed-form solution for the mean and Reynolds stresses. This will be called Method I. Since all the difficulty is caused by the square-root sign, the other time-averaging method (see Rodi 1975; Acrivlellis 1977) logically involves squaring (2) and (3) before combining them to obtain the instantaneous-response equation. This equation is then time averaged and an exact solution for the mean velocities and Reynolds stresses obtained. These equations are expressed in terms of the time average of the voltage squares as against the mean and r.m.s. voltages of Method I. The second method described will be referred to as Method II.

The usual criticism of Method I is as follows. The method is limited to the investigation of flow fields of low turbulence intensity because of the series expansion and consequent truncation of the response equation. In extreme cases, the expansion itself is mathematically invalid. Several authors such as Guitton (1968) and Heskestad (1965), have attempted to correct the results of this method by including higher-order terms. They measured some of these terms and made simplifying assumptions regarding the other terms to arrive at a correction formula. However, such corrections are not universal and their validity is doubtful. Method II has come to be accepted as a better approach because the time-averaged response expression is provided without neglecting any terms. However, this method too is not without criticism. Since the expressions are based on squared voltages, an additional 'squarer' is required in the measurement system in the case of analog data analysis. The other point of contention is that the mean and the fluctuating velocity fields cannot be separated from each other in the resulting equation. The implication is that the mean field be either measured separately by another device or recourse be taken to Method I. Sampath, Ganesan & Gowda (1982) used the Pitot-static probe to obtain the mean velocities. This is not advantageous in view of the time factor, limitations of the probe at low speeds and the correction required for the mean velocity due to turbulence. Furthermore, the advantage obtained through the use of a hot wire is forfeited. Rodi (1975) developed a hybrid version of Methods I and II, in which the mean velocities are determined from the mean voltage signal and the fluctuating velocity components from the squared signal. This procedure not only introduces approximation at this stage but also requires additional measurements of the mean voltages as well and hence it is time consuming. The mean velocities in a three-dimensional flow field can also be determined using an inclined hot wire following the method outlined by Moussa & Eskinazi (1975). However, this would require extensive calibrations and hence, obviously, much more time. It is also thought that as the mean and fluctuating fields are coupled, this method is more suited for highly turbulent flow fields.

Attempts have been made by Rodi and Acrivlellis to compare these two methods of turbulence measurements in flow fields of different turbulence intensities. Rodi evaluated them in the developed region of a free, round, air jet. From his experiments, Rodi concluded that his results obtained using Method II were more reliable and consistent than those obtained by Method I. However, his measurements and those of Wygnanski & Fiedler (1969), both using Method I, showed considerable difference and he attributed this to the thermal-wake interference of the X-wire used by the latter. Rodi's results for Method II were 10–15 % higher than for Method I. However, the results of Sampath, Ganesan & Gowda (1983) for shear stress, also obtained in the free jet using Method II, were lower than that obtained by Wygnanski & Fiedler using Method I. In other words, if the results of Rodi and Sampath *et al.* for the shear stress obtained by Method II, were compared they would show a discrepancy of anywhere between 20–25 % (Sampath *et al.* 1983). It should be pointed out that Rodi has not compared these two methods on a common basis since his approach to solving by Method II required additional and extensive calibrations, which were not used for computation by Method I.

Acrivlellis evaluated these methods in a fully developed turbulent pipe flow. He first gives a correct system of equations to determine the mean velocity and Reynolds stresses by Method II (his equations (15), (18), (19) and (20)). At a later stage, in order to overcome the disadvantage of using a 'squarer' and to separate the mean and fluctuating flow fields, he arrives at yet another, though incorrect, system of equations (his equations (29)–(33)), as was pointed out by Bartenwerfer (1979). Since Acrivlellis' results were based on incorrect equations no conclusions could be drawn regarding the comparison in the low-turbulence case.

From the foregoing discussion, it can be seen that attempts to compare the methods have not been carried out properly and the comparison is incomplete in many aspects. The comparison was not proper because different calibration procedures were used and an uncertainty analysis was not included in the final analysis to conclude whether the differences, if any, were significant. What is lacking in the comparison is an independent standard with which to compare these methods on a common basis in order to decide without doubt which method is more suitable and under what circumstances. The comparison is not complete for the following reasons. The effect of turbulence intensity needs to be studied thoroughly for this limits the use of these methods. At present there is no conclusive study outlining the limitation and use of these methods. Further, there is a need to evaluate the effects of measurement errors in voltages and uncertainty in the calibration constants. This would help in determining the accuracy in calibration parameters required to obtain the desired uncertainty in the computed results of the two methods.

It is, however, not feasible to carry out an experimental investigation to compare these two methods taking care to satisfy all the above mentioned conditions. For example, it is not experimentally possible to generate data to serve as a standard for comparison with another measuring device since each device has its own limitations and associated measurement errors. At present, however, the hot wires are still believed to be the most reliable device for turbulence measurement. This precludes the availability of an experimental standard. Further, many of the factors such as turbulence intensity and errors in measurements are not easily controlled in a laboratory situation. Hence, it is not easy to study their effects. Therefore, the purpose of this study is to establish a suitable numerical approach to study the problem elucidated. The scheme would include the generation of a pseudo standard for comparing the two methods. Items of particular interest are an estimate of the range of validity of the methods with regards to turbulence intensity, an estimate of the truncation errors in Method I, the effects of measurement errors and uncertainty in calibration constants. This information may possibly be useful in explaining some of the discrepancies found in the experimental comparison of these methods by other authors. Such a study is believed to provide a useful guideline for turbulence measurement with hot wires. Experiments would be in a better position to choose a method depending on their requirements.

A statistical approach based on the Monte Carlo simulation technique (see Brown 1956) is adopted here. This approach was used by Swaminathan *et al.* (1984) to carry out some studies of the calibration of hot wires. The design of the numerical

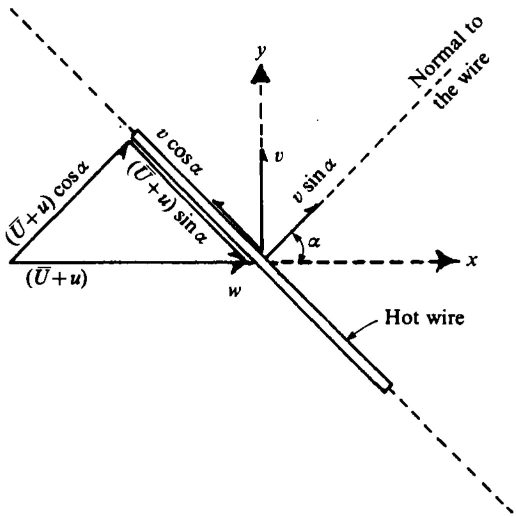

FIGURE 1. Schematic of wire-oriented velocity components.

experiment is based on the response equation alluded to earlier and the generation of pseudo turbulence using random-number generators. After assigning typical values to the mean velocity and calibration constants, the pseudo turbulent field is used to generate the voltages with the assumed response equation. Using random-number generators the voltages generated and the calibration constants are again perturbed about their mean values with preset standard deviations. Use of the perturbed values in the systems of equations for Methods I and II leads to the corresponding solutions for mean and fluctuating velocity fields. This can be compared with the pseudo field generated. The whole experiment is repeated over a large number of trials to obtain an ensemble average of the effects required.

2. Basic equations

The present analysis is restricted to one-dimensional mean flow. The turbulence is however three-dimensional in nature. The hot-wire axis is assumed to be aligned with the mean flow direction in order to simplify the problem. The instantaneous velocity vector is split into components with respect to the wire-oriented coordinate system (see figure 1). It is to be noted that the instantaneous velocity is composed of the mean velocity ($\bar{U}$) in the x-direction and the fluctuating velocity components and (u, v and w) in the x-, y- and z-directions respectively. In figure 1, the hot wire is located in the (x, y)-plane with the normal of the wire forming a yaw angle of α with the mean flow direction, which is along the x-axis. From the geometry it can be seen that

$$\left.\begin{aligned} U_{\mathrm{N}} &= (\bar{U}+u)\cos\alpha + v\sin\alpha, \\ U_{\mathrm{T}} &= -(\bar{U}+u)\sin\alpha + v\cos\alpha, \\ U_{\mathrm{BN}} &= w. \end{aligned}\right\} \tag{4}$$

Introducing (4) into (3), the effective cooling velocity is obtained as

$$U_{\mathrm{eff}} = [\{(\bar{U}+u)\cos\alpha + v\sin\alpha\}^2 + h^2w^2 + k^2\{v\cos\alpha - (\bar{U}+u)\sin\alpha\}^2]^{\frac{1}{2}}. \tag{5}$$

The equation which describes the response of the hot wire to the assumed turbulent flow field is then obtained by substituting (5) into (2) to get

$$\frac{E_{xy}(\alpha)}{s} = [\{(\bar{U}+u)\cos\alpha+v\sin\alpha\}^2+h^2w^2+k^2\{v\cos\alpha-(\bar{U}+u)\sin\alpha\}^2]^{\frac{1}{2}}. \tag{6}$$

Here, $E_{xy}(\alpha)$ is the response of the hot-wire in the (x,y)-plane at a yaw angle α. It is a simple matter to obtain the corresponding expression for the hot-wire response in the (x,z)-plane by replacing v by w and w by v in (6). Now, (6) is time averaged to derive expressions for the mean velocity and the Reynolds stresses. As indicated earlier, to obtain the equations for Method I, the right-hand side of (6) is expanded in a series and the resulting equation is time averaged to acquire the response of the hot wire. For Method II, the equation is first squared and then time averaged. As many equations as the required number of unknowns are generated by orienting the hot wire at different yaw angles in different planes and using the time-averaged response equation. The expression for the mean velocity $\bar{U}$ and the Reynolds stresses $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ and $\overline{uv}$ will be indicated below for both the methods. The details of deriving these equations can be found in Rodi (1975) and Acrivlellis (1977).

Method I.

Mean Velocity:

$$\frac{\bar{E}_{xy}(\alpha=0)}{s} = \bar{U}\left(1+\frac{k^2\overline{v^2}}{2\bar{U}^2}+\frac{h^2\overline{w^2}}{2\bar{U}^2}+O^3\right);$$

Reynolds Stresses:

$$\left.\begin{aligned}
\overline{u^2} &= \frac{\overline{e^2_{xy}}(\alpha=0)}{s^2},\\
\overline{v^2} &= \frac{\overline{e^2_{xy}}(\alpha=45)+\overline{e^2_{xy}}(\alpha=-45)-\overline{e^2_{xy}}(\alpha=0)\{1+k^2\}}{s^2\{1-3k^2\}},\\
\overline{w^2} &= \frac{\overline{e^2_{xz}}(\alpha=45)+\overline{e^2_{xz}}(\alpha=-45)-\overline{e^2_{xy}}(\alpha=0)\{1+k^2\}}{s^2\{1-3k^2\}},\\
\overline{uv} &= \frac{\overline{e^2_{xy}}(\alpha=45)-\overline{e^2_{xy}}(\alpha=-45)}{2s^2\{1-k^2\}}.
\end{aligned}\right\} \tag{7}$$

Here $\overline{e^2_{xy}}(\alpha=0), \overline{e^2_{xy}}(\alpha=45)$ etc. are the mean-square voltages of the fluctuating component in the planes and at the yaw orientations indicated. The above systems of equations have been derived on the assumption that third- and higher-order correlations are negligible compared with the second-order correlations and $k^4=0$. From the system of equations it can be seen that six measurements are to be made: four in the (x,y)-plane consisting of a mean voltage at $\alpha=0°$ and three mean-square values of the fluctuating voltage at $\alpha=0°$, $45°$ and $-45°$, and two mean-square values at $\alpha=+45°$ and $-45°$ in the (x,z)-plane. The expression for the cross correlation $\overline{uw}$ can also be written in terms of these measurements. It has not been included in this paper.

Method II.

The system of equations for Method II for the same hot-wire orientations in the corresponding planes is

$$\left.\begin{aligned} \{\bar{U}^2+\overline{u^2}\}+k^2\overline{v^2}+h^2\overline{w^2} &= \frac{\overline{E^2_{xy}}(\alpha=0)}{s^2}, \\ \{\bar{U}^2+\overline{u^2}\}+\left[\frac{2h^2}{1+k^2}\right]\overline{v^2}+\overline{w^2} &= \frac{\overline{E^2_{xz}}(\alpha=45)+\overline{E^2_{\xi\zeta}}(\alpha=-45)}{\{1+k^2\}s^2}, \\ \{\bar{U}^2+\overline{u^2}\}+\overline{v^2}+\left[\frac{2h^2}{1+k^2}\right]\overline{w^2} &= \frac{\overline{E^2_{xy}}(\alpha=45)+\overline{E^2_{xy}}(\alpha=-45)}{\{1+k^2\}s^2}, \\ \overline{uv} &= \frac{\overline{E^2_{xy}}(\alpha=45)-\overline{E^2_{xy}}(\alpha=-45)}{2s^2\{1-k^2\}}, \end{aligned}\right\} \quad (8)$$

where $\overline{E^2_{xy}}(\alpha = 0)$, $\overline{E^2_{xy}}(\alpha = 45)$ etc. are the mean squares of the hot-wire instantaneous output voltage in the respective planes and at the orientations indicated. This system requires measurement of five squared voltages at a point to solve for the unknowns $(\bar{U}^2+\overline{u^2})$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$ and $\overline{uw}$. The expression for $\overline{uw}$ has not been included. It is obvious that one more measurement is required to separate the terms $\bar{U}^2$ and $\overline{u^2}$. This is the inherent problem in this method. It is to be noted here that the expressions are in terms of the time average of the squared voltage as against the mean-square values of the fluctuating voltage and the mean voltage in Method I.

3. Monte Carlo testing procedure

The procedure adopted is essentially that followed by Krutchkoff (1967) for evaluating the direct and indirect regression methods. The procedure followed here can be divided into two major parts. The first part deals with the generation of pseudo turbulence using random-number generators, calculation of the average stress field from the generated turbulence and the use of (6) to obtain the voltage information. The second part deals with the comparison of the two methods and a study of the effects of measurement errors. The voltage information obtained in the first part forms an input to the second part. The stress field obtained in the first part serves as a standard to compare the Reynolds stresses obtained by using the systems of equations for Method I and Method II.

At the outset, the constants occurring in (6) viz. s, k, h, and the mean velocity ($\bar{U}$) are assigned typical values without any loss of generality (see Krutchkoff). Using the random-number generator a set of u-, v- and w-values are now picked up with preassigned means and standard deviations. The mean was, of course, set at zero and the standard deviations were varied to change the pseudo turbulence levels. The set of u-, v- and w-values were used in (6) and the corresponding equation in the (x, z)-plane at different orientations mentioned earlier, to obtain five values of the instantaneous voltages $E_{xy}(\alpha = 0)$, $E_{xy}(\alpha = +45)$, $E_{xy}(\alpha = -45)$, $E_{xz}(\alpha = +45)$ and $E_{xz}(\alpha = -45)$. From u, v and w the instantaneous values of the stresses u^2, v^2, w^2 and uv-values can be computed. The random-number generator is again used to choose another set of (u, v, w) and the above procedure is repeated over a large number of trials. The ensemble averages of the Reynolds stress $\overline{u^2_0}$, $\overline{v^2_0}$, $\overline{w^2_0}$ and $\overline{uv}_0$ are found using the instantaneous voltage values, obtained over a large number of trials. It is a simple matter to determine the ensemble averages of the voltages $\overline{E^2}$, $\bar{E}$ and $\overline{e^2}$ for different orientations in different planes. Here, the ergodic hypothesis is assumed to apply. Thus, the standard values for the Reynolds stresses and the corresponding time-averaged voltage values are generated.

The second part consists of introducing errors into the values of $\overline{U}$, $\overline{E^2}$, $\overline{E}$ and $\overline{e^2}$, the yaw angle α and the calibration constants s and k. It should be pointed out that h is assumed error-free throughout the analysis. Random-number generators are then used to choose errors in these quantities with a preassigned standard deviation. Using these perturbed values and the systems of equations for Method I and Method II, the mean velocity and the Reynolds stresses are obtained for both the methods. These values are then compared with the standard values generated for the stresses in part I. The procedure is repeated over a large number of trials to estimate the effects of errors in the aforementioned quantities and to compare the two methods. The turbulence levels can be varied by changing the standard deviations in u, v and w keeping $\overline{U}$ fixed. The whole procedure is then repeated to study their effects. The comparisons are based on the normalized standard deviations in the Reynolds stresses expressed as percentages, viz. $[(\overline{u_{\mathrm{I}}^2}-\overline{u_0^2})\times 100]/\overline{u_0^2}$, etc. Here $\overline{u_{\mathrm{I}}^2}$, etc. are the stresses obtained using the corresponding method and $\overline{u_0^2}$, etc. are the standard values generated.

4. Numerical experiments

The numerical experiments were carried out on an IBM 3031 computer based on the logic indicated. The IBM package 'RANDU' was used for generating the random numbers. The numerical experiment consisted of testing the random-number generator, determination of the number of trials required to obtain reliable results and finally carrying out the required experiments.

The random-number generator produces numbers, the mean and standard deviation of which differ slightly from the preset values. This would introduce errors in the final result. Hence the random numbers generated were modified by adding a fixed quantity to obtain (the desired mean exactly). This fixed quantity was the difference between the desired mean and the actual mean of the random numbers obtained. The numbers thus obtained were multiplied by a constant factor such that the required standard deviation was also exactly obtained. This constant factor was equal to the ratio of the desired standard deviation to the actual standard deviation obtained after the first modification of the random numbers. Further, one more condition was stipulated on u and v to obtain a non-zero cross-product term $\overline{uv}$. The condition was: whenever u was positive, v was assigned a negative value and vice versa. This condition amounted to assuming the sign of the slope of the mean velocity at the hypothetical point of 'measurement' based on the mixing-length model (see Schlichting 1968). These operations can be shown not to affect the characteristics of the generated random numbers (see Hamming 1962). The logic indicated was incorporated in the software.

The second step was to determine the number of trials required to generate a reliable pseudo turbulence field. The criterion chosen was that the ensemble average of the shear stress $(\overline{uv}_0)$, computed from the field generated, be independent of the number of trials. Since modification of the random numbers, described earlier, yielded a constant value for $\overline{u_0^2}$, $\overline{v_0^2}$ and $\overline{w_0^2}$ independent of the number of trials, the criterion was fixed based on the $\overline{uv}_0$ value. The effect of the seed value, required to initiate the random-number generator, was also investigated. To determine the number of trials required to generate a reliable pseudo turbulence field, the mean velocity and the constants occurring in (6) needed to be prescribed. The typical values assigned were $\overline{U} = 25$ m/s, $s = 0.3$ volts/m/s, $\alpha = \pm 45°$, $k = 0.2$ and $h = 1.0$. The standard deviations of the errors in s, α, k, $\overline{U}$ and the voltages were set at zero. The standard deviations of the fluctuating velocity components u, v and w were set at 10% of the mean velocity. Keeping these factors constant, the number of trials was varied between 5 and 40000 to determine the number of trials at which $(\overline{uv}_0)$ attained a constant value. The number of trials required for the second part, when measurement errors were introduced, was arbitrarily set at 5000 trials. From previous work (see Swaminathan, Rankin & Sridhar 1984) this number is considered to be large enough to yield reliable results.

The numerical experiments were then carried out to estimate the approximation involved in Method I and its variation with turbulence intensity, to study the effects of measurement errors in s, α, k, $\overline{U}$, $\overline{E^2}$ and $\overline{e^2}$, and finally, to evaluate the two methods. To estimate the truncation error in Method I, the standard deviations in measurement errors were set at zero. Under these conditions, Method II should yield the same results for the Reynolds stresses as the standard since no approximations are involved. However, Method I is likely to produce deviations from the stress field generated due to truncation and the consequent approximation involved. The difference in the Reynolds stresses between these two methods would then give an estimate of this truncation error in Method I. The effect of truncation on the mean velocity, determined using Method I, is described below. From the expression for mean velocity in (7), we can write

$$\frac{\overline{E}_{xy}(\alpha = 0)}{s\overline{U}} = 1 + \frac{k^2\overline{v^2}}{2\overline{U}^2} + \frac{h^2\overline{w^2}}{2\overline{U}^2} + O^3. \tag{9}$$

The product $s\overline{U}$ can be determined from the assumed values of s and $\overline{U}$. This represents the mean voltage output $\overline{E}_0$ of the hot wire at $\alpha = 0$ and with no turbulence present in the stream. $\overline{E}_{xy}(\alpha = 0)$ represents the mean voltage generated using random numbers when turbulence is present. From the turbulence field generated the quantities $\overline{v^2}$ and $\overline{w^2}$ are known. Hence the right-hand side of the above equation can be independently evaluated to the third order of accuracy and is the commonly used turbulence factor f. If the right-hand side is evaluated without any truncation it may be considered as an exact factor f_{exact}. In physical experiments, f_{exact} cannot be obtained because $\overline{U}$ is not known *a priori* and because of the difficulty in measuring the higher-order terms. In this numerical experiment, however, f_{exact} can be determined from the values of s, $\overline{U}$ and $\overline{E}_{xy}$. One may rewrite (9) as

$$f_{\text{exact}} = f + O^3. \tag{10}$$

The variations of f and f_{exact} with turbulence intensity were studied by changing the levels between 0 and 50%, keeping all the other factors constant.

To study the effects of measurement errors in $\overline{U}$, s, k, and the voltages two separate experiments were carried out. For this exercise the yaw angle α was assumed error-free. In the first, a uniform error of 1% was introduced in all the quantities $\overline{U}$, s, k, $\overline{E^2}$ and $\overline{e^2}$ and the normalized standard deviations in the Reynolds stresses were obtained. The error in only one of the above quantities was then held at 1% and the errors in other quantities were set at zero. The experiment was repeated to estimate the individual contribution of the various quantities to the total error. This experiment was carried out for different turbulence intensities in the range 10–50%. In the second experiment, the errors in these quantities were set at the worst possible combination based on past experience. The standard deviations of the errors chosen

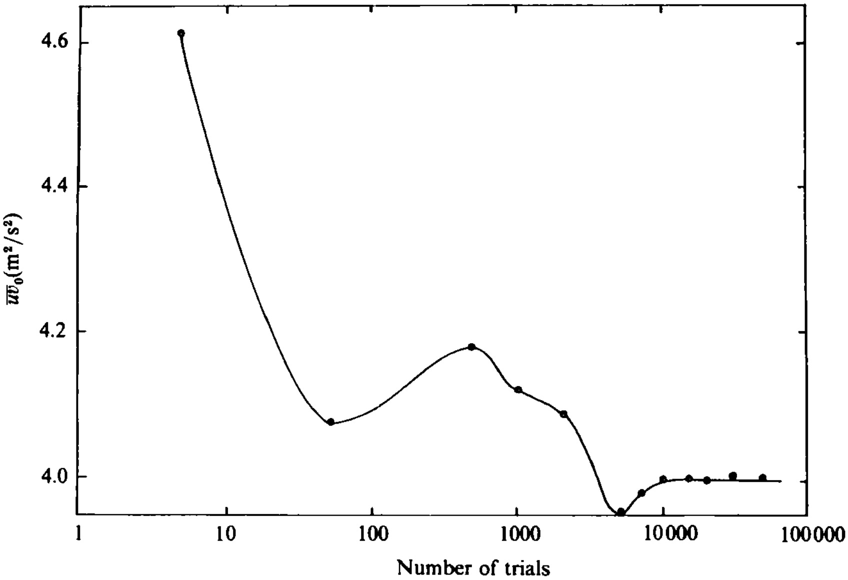

FIGURE 2. Effect of number of trials on $\overline{uv}_0$. Standard deviation in u, v and $w = 2.5$ m/s, $s = 0.3$ volts/m/s, $k = 0.2$, $h = 1.0$ and $\overline{U} = 25$ m/s.

were 1 % in $\overline{U}$, 0.5 % in $\overline{E^2}$ and $\overline{e^2}$, 10 % in s and 20 % in k. For this combination of errors, the normalized standard deviations in the Reynolds stresses were obtained for both the methods. Then, the error in each quantity was varied between zero and the limit chosen while keeping the errors in the other quantities fixed at the values chosen. In order to estimate the effect of error in the yaw angle, the errors in the other quantities were set at zero. This was done to isolate the effects of errors in α. Only fixed types of errors in α were treated. This is justified from a practical point of view. The errors in α were varied in the range 0–2 %. The experiments on the effect of measurement errors were carried out for only two values of turbulence intensity, 10 % and 50 %.

5. Results and discussion

5.1. *Number of trials required*

The number of trials required to generate a reliable pseudo turbulence field is based on the result indicated in figure 2. This figure gives the variation of $\overline{uv}_0$ with the number of trials. It is seen that the Reynolds shear stress attains a constant value beyond 10000 trials. However, the number of trials was set at 15000 to allow a factor of safety. This number was held constant for all subsequent experiments. Holding this constant, the seed value required to initiate the random-number generator was varied to ensure that no significant changes were observed.

5.2. *Truncation errors in Method* I

The results of the variation of f and f_{exact} with turbulence intensity is given in figure 3. The percentage variation between the two is in the range 0–1.5 % as the intensity is changed from 0 to 50 %, indicating that the second-order corrections for the mean velocity yields accurate results for the range studied. The common assumption that $f = 1$ is seen to be justified only up to a 10 % turbulence intensity beyond which the

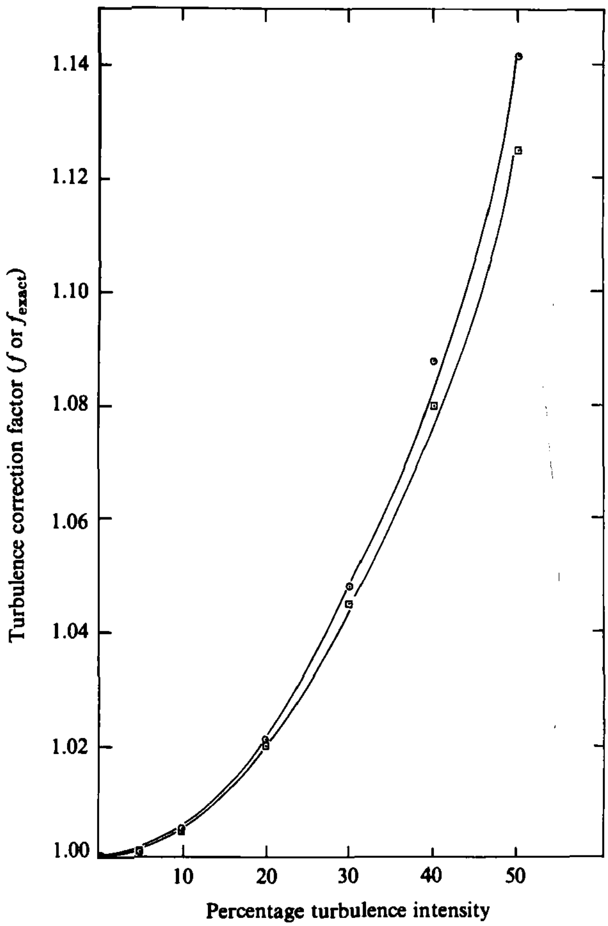

FIGURE 3. Variation of turbulence correction factor for mean velocity with turbulence intensity. ⊙, f_{exact}; ⊡, $f = 1 + (\overline{v^2}k^2 + \overline{w^2}h^2)/2\overline{U}^2$.

errors in the mean velocity are significant. The variation of the errors introduced in the Reynolds stresses, calculated using Method I, due to the truncation of the series expansion with turbulence intensity can be seen in figure 4 for the values of s, k and $\overline{U}$ chosen. As expected Method II produces insignificant errors in $\overline{u^2}$, $\overline{v^2}$ and $\overline{uv}$. It is believed that these insignificant errors are due to roundoff errors. From this figure, it is seen that beyond 20 % turbulence intensity, the errors due to truncation increase considerably. Errors in $\overline{v^2}$ and $\overline{uv}$ due to truncation are seen to be higher compared with errors in $\overline{u^2}$. The errors in $\overline{w^2}$ and $\overline{v^2}$ are almost identical and hence have not been included in the figure. For turbulence intensities below 20 %, the truncation errors in Reynolds stresses can be held below 10 % for the values of the constants chosen. By repeating the numerical experiment, it is a simple matter to estimate the truncation error should the values of s, k and $\overline{U}$ change.

5.3. *Effects of measurement errors*

The results obtained by choosing a uniform 1 % measurement error in the quantities s, k, $\overline{U}$, $\overline{E^2}$ and $\overline{e^2}$ and zero error in α for the cases of low and high turbulence

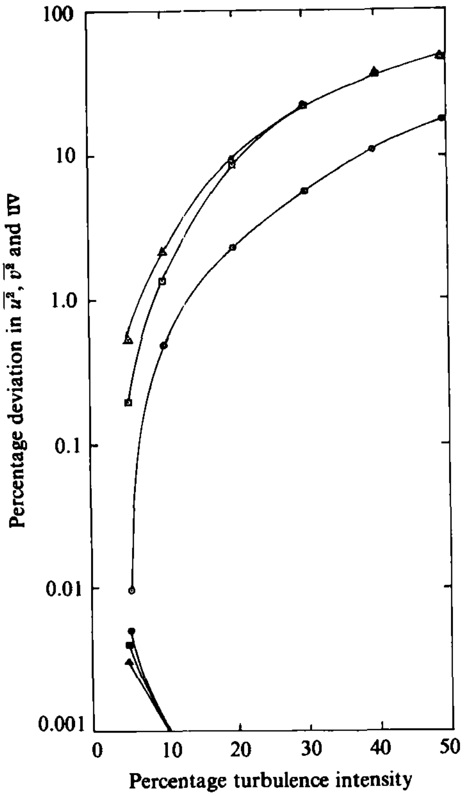

FIGURE 4. Effect of turbulence intensity (errors in s, k, $\bar{U}$, $\overline{E^2}$ and $\overline{e^2}$ = 0%). Method 1: ⊙, $\overline{u^2}$; ⊡, $\overline{v^2}$: △, $\overline{uv}$. Method 2: ●, $\overline{u^2}$; ■, $\overline{v^2}$; ▲, $\overline{uv}$.

Remarks	Method I: Percentage standard deviation in				Method II: Percentage standard deviation in			
	$\overline{u^2}$	$\overline{v^2}$	$\overline{w^2}$	$\overline{uv}$	$\overline{u^2}$	$\overline{v^2}$	$\overline{w^2}$	$\overline{uv}$
1% error in s k, $\bar{U}$, $\overline{E^2}$ and $\overline{e^2}$	2.3	3.3	3.4	3.2	337.0	67.0	104.0	61.0
1% error in s	2.0	2.4	2.7	2.9	202.0	2.0	2.0	2.0
1% error in k	0.5	1.4	1.9	2.2	4.2	4.1	4.1	0.1
1% error in $\bar{U}$	0.5	1.4	1.9	2.2	200.0	0.0	0.0	0.0
1% error in $\overline{E^2}$	0.5	1.4	1.9	2.2	182.0	66.0	104.0	61.0
1% error in $\overline{e^2}$	1.1	2.6	2.8	2.5	0	0	0	0
0% error in all of the quantities	0.5	1.4	1.9	2.2	0	0	0	0

TABLE 1. Contribution to errors in Reynolds stresses by various parameters (10% turbulence intensity)

Remarks	Method I: Percentage standard deviation in				Method II: Percentage standard deviation in			
	$\overline{u^2}$	$\overline{v^2}$	$\overline{w^2}$	$\overline{uv}$	$\overline{u^2}$	$\overline{v^2}$	$\overline{w^2}$	$\overline{uv}$
1 % error in s k, $\overline{U}$, $\overline{E^2}$ and $\overline{e^2}$	18.0	46.7	45.2	48.8	17.3	4.8	7.8	5.1
1 % error in s	18.0	46.7	45.2	48.8	10.0	2.0	2.0	2.0
1 % error in k	18.0	46.7	45.2	48.8	0.1	0.2	0.2	0.1
1 % error in $\overline{U}$	18.0	46.7	45.2	48.8	8.0	0.0	0.0	0.0
1 % error in $\overline{E^2}$	18.0	46.7	45.2	48.8	11.6	4.4	7.5	4.7
1 % error in $\overline{e^2}$	18.0	46.7	45.2	48.8	0	0	0	0
0 % error in all of the quantities	18.0	46.7	45.2	48.8	0	0	0	0

TABLE 2. Contribution to errors in Reynolds stresses by various parameters (50 % turbulence intensity)

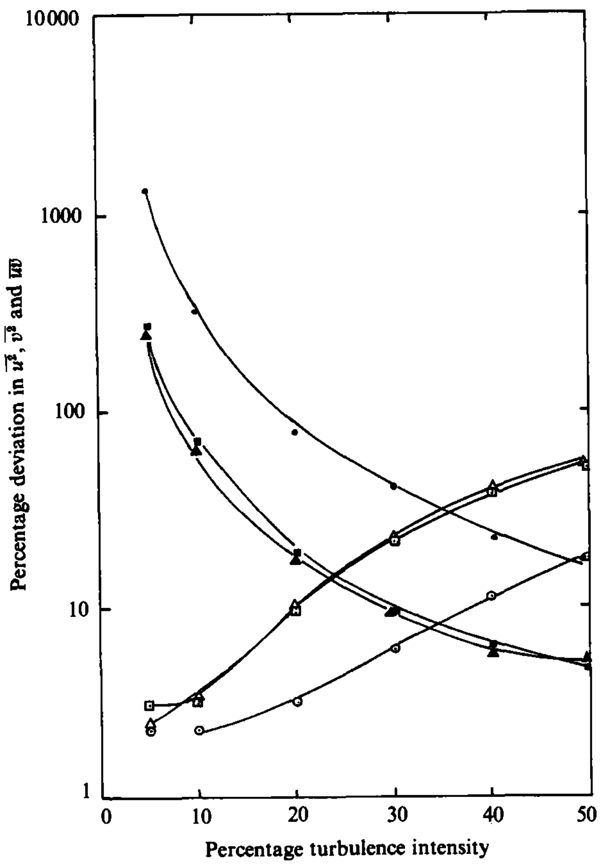

FIGURE 5. Effect of turbulence intensity (errors in s, k, $\overline{U}$, $\overline{E^2}$ and $\overline{e^2} = 1$ %). Method 1: ⊙, $\overline{u^2}$; ⊡, $\overline{v^2}$; △, $\overline{uv}$. Method 2: ●, $\overline{u^2}$; ■, $\overline{v^2}$; ▲, $\overline{uv}$.

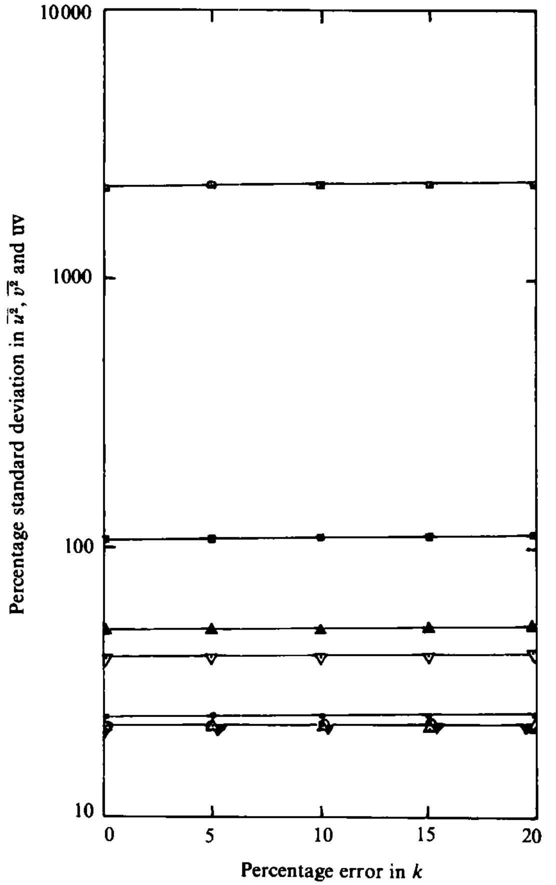

FIGURE 6. Effect of variation of errors in k on Reynolds stresses (errors in $s = 10\%$, $\overline{U} = 1\%$, $\overline{E^2} = 0.5\%$ and $\overline{e^2} = 0.5\%$). Low-turbulence case: ⊙, $\overline{u^2}$, Method 1; ⊡, $\overline{u^2}$, Method 2; △, $\overline{uv}$, Method 1; ▽, $\overline{uv}$, Method 2. High-turbulence case: ●, $\overline{u^2}$, Method 1; ■, $\overline{u^2}$, Method 2; ▲, $\overline{uv}$, Method 1; ▼, $\overline{uv}$, Method 2.

intensities of 10% and 50% are given in tables 1 and 2. Comparison of the tables indicates that Method II yields lower errors in the ensemble-averaged variables at high turbulence intensity and Method I at low levels. From table 1 for low turbulence levels it is seen that for Method I a 1% error in all the measurable quantities produces errors in Reynolds stresses which are comparable to the truncation errors (compare values in row one to the values in row seven of table 1). At low turbulence levels, Method II yields considerable errors, especially in $\overline{u^2}$. For this case it is seen that Method I is preferable. The system of equations for Method II is ill-conditioned for the case of low turbulence and hence the results are sensitive to measurement errors. The major contributing factors to the overall error are the errors in s, the voltages and mean velocity in the case of Method II. Table 2 gives the results for the high-turbulence case. For Method I, it is seen that the truncation errors are so high (values in the last row of the table) that the errors introduced in the various quantities have no effect on the final result. Method II, however, yields reasonable overall results for the high-turbulence case. Here again the major contributing factors are the errors in s, $\overline{U}$ and the voltages. It is consistently seen that effects of

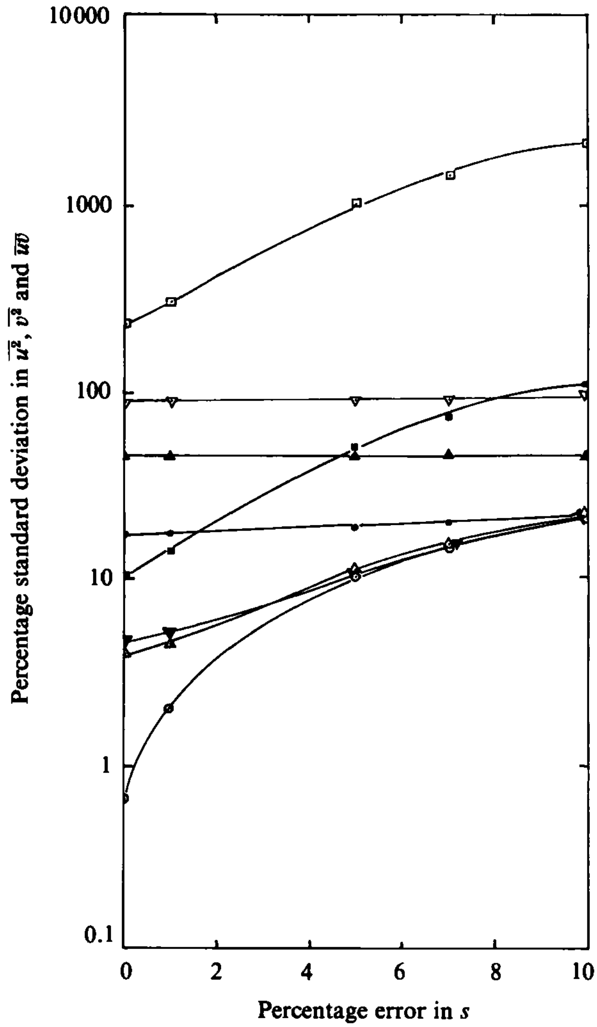

FIGURE 7. Effect of variation of errors in s on Reynolds stresses (errors in $k = 20\,\%$, $\overline{U} = 1\,\%$, $\overline{E^2} = 0.5\,\%$ and $\overline{e^2} = 0.5\,\%$). Low-turbulence case: ⊙, $\overline{u^2}$, Method 1; ⊡, $\overline{u^2}$, Method 2; △, $\overline{uv}$, Method 1; ▽, $\overline{uv}$, Method 2. High-turbulence case: ●, $\overline{u^2}$, Method 1; ■, $\overline{u^2}$, Method 2; ▲, $\overline{uv}$, Method 1; ▼, $\overline{uv}$, Method 2.

errors in k are insignificant and that, for Method II, measurement of $\overline{U}$ significantly affects the determination of $\overline{u^2}$ especially for the low-turbulence case.

The effect of turbulence intensity for the case of 1 % uniform error in all the quantities is shown in figure 5. It can be observed that for Method I the error in $\overline{u^2}$ is lower compared with $\overline{v^2}$ and $\overline{uv}$ throughout the range of turbulence intensity studied. The opposite is true for Method II. Below 20 % turbulence level, Method I is recommended and above 40–50 % Method II is recommended. It should be pointed out that the values of standard deviation in errors chosen may be unrealistic. Should a more reliable estimate be available, it is a simple matter to generate the results and obtain the range of validity of these methods with regards to the turbulence intensity. If the standard deviations of the errors in the measured quantities are

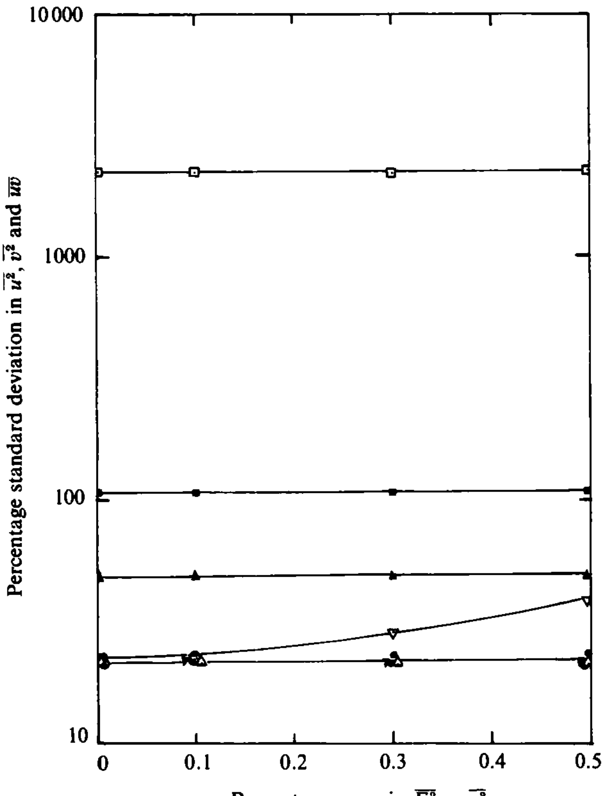

FIGURE 8. Effect of variation of errors in $\overline{E^2}$ or $\overline{e^2}$ on Reynolds stresses (errors in $s = 10\%$, $\overline{U} = 1\%$ and $k = 20\%$). Low-turbulence case: ⊙, $\overline{u^2}$, Method 1; ⊡, $\overline{u^2}$, Method 2; △, $\overline{uv}$, Method 1; ▽, $\overline{uv}$, Method 2. High-turbulence case: ●, $\overline{u^2}$, Method 1; ■, $\overline{u^2}$, Method 2; ▲, $\overline{uv}$, Method 1; ▼, $\overline{uv}$, Method 2.

known, then the Monte Carlo procedure, described earlier, can be used to determine the uncertainties in the measured quantities. The method suggested here, which is an extension of that proposed by Moffat (1982) for determining the uncertainties in the measurements, includes the use of a random-number generator to simulate random errors within the limits specified and averaging the effects over large number of trials. It is believed that this would yield a more reliable estimate of the uncertainties in the measured quantities. The logic is simple and it should pose no problem to incorporate it in the data-reduction routine.

In the second experiment to determine the effects of errors, the standard deviations of the errors in s, k, $\overline{U}$ and voltages were set at 10, 20, 1 and 0.5% respectively with the error in α being set at zero. These values correspond to the worst possible combination of the errors based on past experience. The results obtained by varying the errors in k between 0–20% of the mean value of 0.2, keeping all the other errors at the chosen value, are shown in figure 6 for the cases of both low and high turbulence intensities. It is seen for the values chosen, that the errors in k have an insignificant effect on the final results. The error values obtained in Reynolds stresses are large indicating that the errors assigned are also large. The effects of the variation of errors in s and voltages are shown in figures 7 and 8 respectively. From these plots, only the errors in s seem to have a significant effect on the accurate determination of the Reynolds stresses. However, no particular trend can be discerned from this

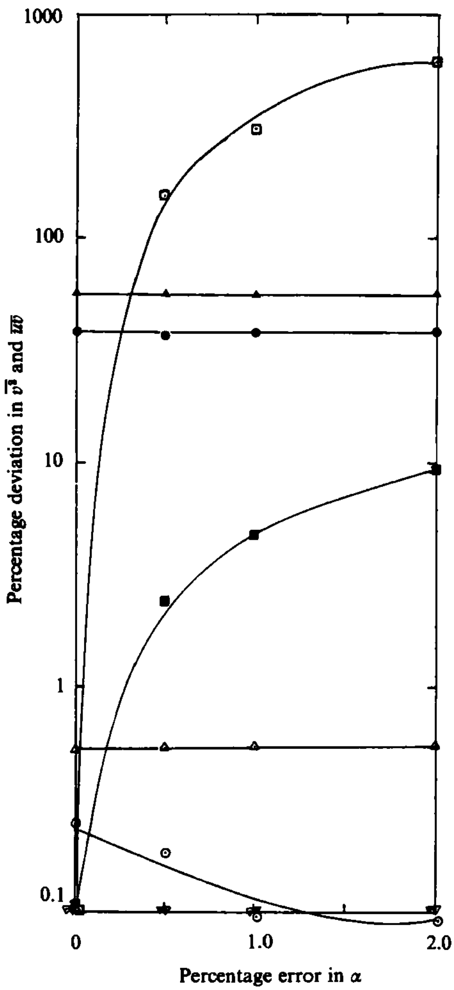

FIGURE 9. Effect of variation of errors in α on Reynolds stresses (error in s, $\overline{U}$, $\overline{E^2}$ and $\overline{e^2} = 0\,\%$). Low-turbulence case: ⊙, $\overline{v^2}$, Method 1; ⊡, $\overline{v^2}$, Method 2; △, $\overline{uv}$, Method 1; ▽, $\overline{uv}$, Method 2. High-turbulence case: ●, $\overline{v^2}$, Method 1; ■, $\overline{v^2}$, Method 2; ▲, $\overline{uv}$, Method 1; ▼, $\overline{uv}$, Method 2.

plot. For the combination of errors prescribed here, it can be seen from figures 5–7 that Method I is recommended for both the low- and the high-turbulence-level cases. A more rigorous procedure would be to repeat the numerical experiment for different combinations of the errors in s, $\overline{U}$, k and the voltages and to study the effects of variation of these errors.

The effect of errors in α on the percentage standard deviation in $\overline{v^2}$ and $\overline{uv}$ is shown in figure 9 for the case of zero errors in voltages, s, k and $\overline{U}$. It can be seen that Method I is insensitive to errors in α. The $\overline{v^2}$ parameter is, however, seen to be affected by errors in α for Method II. The effect of an error in α would be to cause an error in the mean and r.m.s. voltages measured. This, in turn, would cause errors in the Reynolds stresses. The truncation error in Method I for the high-turbulence case and the inappropriateness of Method II in the low-turbulence range, should be taken into account while interpreting the data. These errors are large enough to suppress any trends with respect to α.

6. Conclusions

The Monte Carlo testing procedure has been successfully used to simulate turbulence and the corresponding hot-wire response from the assumed analytical expression. This procedure has been used to compare the two methods of obtaining the time-averaged mean velocity and Reynolds stresses. Also included in the study are the estimation of the truncation error in the conventional method, the effects of turbulence intensity and measurement errors on the results. The results indicate that the truncation errors in Method I for Reynolds stresses are less than 10 % when turbulence levels are below 20 %. For higher values of turbulence level, the error increases rapidly. For mean velocity measurement by Method I, the truncation errors are below 1.5 % for turbulence intensities as high as 50 %. The choice of method should be based on both the turbulence levels and the measurement errors expected. If the measurement errors are of the order of 1 % in all the quantities, then Method I is recommended for turbulence levels below 20 % and Method II for levels above 40–50 %. If measurement errors are expected to be high (i.e. of the order 10–20 % in s and k) Method I proves to be relatively better. To obtain accurate results, the hot-wire sensitivity s and the voltages have to be determined as accurately as possible. The error in the yaw sensitivity k has negligible effect on the results obtained. Finally, a rigorous method for uncertainty analysis has also been introduced, which makes use of random-number generators and is based on Monte Carlo testing.

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